

Effective Action in a General Chiral Model: Next to Leading Order Derivative Expansion in the Worldline Method

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We present a formalism to determine the imaginary part of a general chiral model in the derivative expansion. Our formalism is based on the worldline path integral for the covariant current that can be given in an explicit chiral and gauge covariant form. The effective action is then obtained by integrating the covariant current, taking account of the anomaly.

I. INTRODUCTION

When discussing the influence of fermions on the dynamics of some theory, e.g. of the Standard Particle Model (SM) in a cosmology setting, it is mandatory to integrate out the fermions. This procedure in one-loop order already becomes quite involved if arbitrary chiral couplings of space-time dependent outer bosonic fields are considered. The extensive work of Salcedo [1, 2] resulted in an effective action to leading order in covariant derivatives of such fields. Basis is a refined calculation in momentum space, handling of anomalies using the Wess-Zumino-Witten model (WZW), and last but not least, a very practical shorthand notation.

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Such effective actions are quite important in evaluating some models. For example, in ref. [3] Smit discussed a form of 'cold' electroweak baryogenesis at the end of electroweak scale inflation [4] which could very well work if the rephasing invariant

$$J = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin(\delta) = (3.0 \pm 0.3) \times 10^{-5} \quad (1)$$

of the Jarlskog determinant is not accompanied by further suppressions through mass ratios. It was proposed [3] that derivative terms in the effective action that are analytic in the time-dependent masses considered non-perturbatively could be very important in non-equilibrium. Such effects were also observed in ref. [5]. In the work [3] the fourth order derivative result of ref. [1] turned out not to contain CP violation, but the claim was made that higher orders of the imaginary part of the effective action will do.

Worldline methods in first quantized quantum field theory are ideally adapted for calculating effective actions: One considers the propagation of a particle in some space-time dependent background [6], but in x-space path integral formulation [7]. This method [8, 9], also related to the infinite tension limit of String theory [10, 11], was used heavily for the discussion of various effective actions in one-loop [9, 12, 13, 14] and two-loop [15, 16, 17, 18] order. For example, the high order in the inverse mass calculation of ref. [15] could hardly be done with other methods.

The present paper provides a formalism to determine higher order contributions to the imaginary part of the effective action using the worldline formalism. We are concerned with the effective action of a multiplet of N Dirac fermions coupled to an arbitrary matrix-valued set of fields, including a scalar Φ , a pseudoscalar Π , a vector A , a pseudovector B , and an antisymmetric tensor $K_{\mu\nu}$. One peculiar feature of the imaginary part of the effective action is that it cannot be written in a manifest chiral covariant way, due to the presence of the chiral anomaly. One possibility to arrive at a closed expression for the effective action is to abandon manifest chiral covariance as it was done in ref. [19]. The resulting expression is rather complicated and not well suited for higher order calculations. Alternatively, it was proposed in ref. [1] to determine the covariant current for which a manifestly chiral covariant expression exists and to take account of the anomaly when integrating the current to yield the effective action. Following this idea, we present a worldline path integral formulation of the covariant current.

Before we do so, we review the worldline formalism by discussing the derivation of the

real part of the effective action. A single Dirac fermion in the presence of both a scalar and pseudoscalar fields in the context of the worldline formalism was first treated in ref. [20], the inclusion of a pseudovector in ref. [21]. In our discussion of the real part of the effective action we will follow the elegant subsequent work of refs. [19, 22].

Section II contains the derivation of the real part of the effective action, the derivation of the covariant current and the matching procedure to obtain the imaginary part of the effective action. In section III, we briefly reproduce the results in lowest order from ref. [1]. As a novel result we present the imaginary part of the effective action in two dimensions in next to leading order in section IV.

II. EFFECTIVE ACTION

We are concerned with the effective action

$$iW[\Phi, \Pi, A, B, K] = \log \text{Det } i[i\cancel{\partial} - \Phi + i\gamma^5\Pi + \cancel{A} + \gamma^5\cancel{B} + i\gamma^\mu\gamma^\nu K_{\mu\nu}], \quad (2)$$

and its continuation to Euclidean space. The γ matrices remain unaffected by the continuation, but it is useful to introduce the following notation, $(\gamma_E)_j \equiv i\gamma_j$, $(\gamma_E)_4 \equiv \gamma_0$, and $(\gamma_E)_5 \equiv \gamma_5$. After Wick-rotation, $t \rightarrow -it$, one obtains with this new notation

$$\cancel{\partial} \rightarrow i\cancel{\partial}_E, \cancel{A} \rightarrow i\cancel{A}_E, \cancel{B} \rightarrow i\cancel{B}_E, \gamma^\mu\gamma^\nu K_{\mu\nu} \rightarrow -(\gamma_E)_\mu(\gamma_E)_\nu K_{E\mu\nu}. \quad (3)$$

From now on, the E subscript will be suppressed.

The effective action of Eq. (2) now reads

$$-W[\Phi, \Pi, A, B, K] = \log \text{Det } [\mathcal{O}], \quad (4)$$

with the operator \mathcal{O} in momentum space defined by

$$\mathcal{O} \equiv \not{p} - i\Phi(x) - \gamma_5\Pi(x) - \cancel{A}(x) - \gamma_5\cancel{B}(x) + \gamma_\mu\gamma_\nu K_{\mu\nu}. \quad (5)$$

As in ref. [2, 22], the real and imaginary parts of the effective action are analyzed separately

$$-W^+ - iW^- = \log(|\text{Det } [\mathcal{O}]|) + i \arg(\text{Det } [\mathcal{O}]). \quad (6)$$

A perturbative expansion in weak fields [22] shows that graphs with an even number of γ_5 vertices are real, and graphs with an odd number of γ_5 vertices are imaginary. This will prove useful when the behavior of the effective action under complex conjugation is explored later on.

A. Real Part of the Effective Action

Our intention is to obtain a worldline representation for the effective action with manifest chiral and gauge invariance. This is unproblematic for the real part, but it causes certain difficulties for the imaginary part due to the chiral anomaly. In order to familiarize the reader with the worldline method we review the derivation for the real part in four dimensions as it was presented in refs. [19, 22].

1. Construction of a Positive Operator for the Real Part of the Effective Action

In order to use the worldline formalism, one has to rewrite the effective action in terms of a positive operator, thus obtaining

$$W^+ = -\frac{1}{2} \log \text{Det} [\mathcal{O}^\dagger \mathcal{O}]. \quad (7)$$

The problem with this operator is that it contains terms linear in the γ matrices, what makes the transition to a path integral of Grassman fields problematic. One way to avoid this problem is by doubling the fermion system and exchanging the operator \mathcal{O} for a Hermitian operator Σ yielding

$$W^+ = -\frac{1}{2} \log \text{Det} [\mathcal{O}^\dagger \mathcal{O}] = -\frac{1}{4} \log \text{Det} [\Sigma^2], \quad \Sigma \equiv \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O}^\dagger & 0 \end{pmatrix}. \quad (8)$$

Since Σ is Hermitian, one can use the Schwinger integral representation of the logarithm without any restrictions. One obtains

$$W^+ = \frac{1}{4} \int_0^\infty \frac{dT}{T} \text{Tr} \exp(-T\Sigma^2). \quad (9)$$

At this point, it is natural to introduce six 8×8 Hermitian Γ_A matrices. These matrices satisfy $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$, with $A, B = 1..6$ and are defined as

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & i\mathbb{1}_4 \\ -i\mathbb{1}_4 & 0 \end{pmatrix}. \quad (10)$$

For later use we also introduce the equivalent of γ_5 ,

$$\Gamma_7 = -i \prod_{A=1}^6 \Gamma_A = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \quad (11)$$

and Γ_7 anticommutes with all other Γ matrices.

Expressing Σ in terms of these new matrices yields [19],

$$\Sigma = \Gamma_\mu(p_\mu - A_\mu) - \Gamma_6\Phi - \Gamma_5\Pi - i\Gamma_\mu\Gamma_5\Gamma_6B_\mu - i\Gamma_\mu\Gamma_\nu\Gamma_6K_{\mu\nu}. \quad (12)$$

The aim is to turn Eq. (12) into an expression which is manifestly chiral covariant. This can be achieved by changing to a basis in which $i\Gamma_5\Gamma_6$ is diagonal [22] using the following transformation

$$M^{-1}i\Gamma_5\Gamma_6M = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \quad M = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_2 \\ 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \end{pmatrix}. \quad (13)$$

In this basis, Σ takes the form

$$\tilde{\Sigma} = M^{-1}\Sigma M = \begin{pmatrix} \gamma_\mu(p_\mu - A_\mu^L) & \gamma_5(-iH + \frac{1}{2}\gamma_\mu\gamma_\nu K_{\mu\nu}^s) \\ -\gamma_5(-iH^\dagger + \frac{1}{2}\gamma_\mu\gamma_\nu K_{\mu\nu}^{s\dagger}) & \gamma_\mu(p_\mu - A_\mu^R) \end{pmatrix}, \quad (14)$$

which is manifestly chiral covariant. Here $A^L = A + B$, $A^R = A - B$, $H = \Phi - i\Pi$, $K^s = K - i\tilde{K}$ and $\tilde{K}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}K^{\rho\sigma}$ have been defined.

The square of $\tilde{\Sigma}$ constitutes a positive operator which is suitable for the worldline formalism. However, even though this expression contains only even combinations of γ matrices, the coherent state formalism cannot yet be used to transform this expression into a fermionic path integral. In the coherent state formalism, the γ_5 matrices have to be rewritten as a product of the other γ matrices, what would result again in odd combinations. One possible solution of this problem is to enlarge the Clifford space, replacing the γ matrices by Γ matrices

$$\gamma_A \rightarrow \Gamma_A = \gamma_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A \in [1 \dots 5]. \quad (15)$$

The matrix Γ_5 is then independent from the other Γ matrices and the coherent state formalism with six (instead of four) operators can be used. The doubling of the Clifford space inside the trace has to be compensated by a factor $\frac{1}{2}$, such that Eq. (9) reads

$$W^+ = \frac{1}{8} \int_0^\infty \frac{dT}{T} \text{Tr} \exp(-T\hat{\Sigma}^2), \quad (16)$$

and the operator $\hat{\Sigma}^2$ is given by

$$\begin{aligned}\hat{\Sigma}^2 = & (p - \mathcal{A})^2 + \mathcal{H}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}_{\mu\nu} + \frac{i}{2}\Gamma_\mu\Gamma_\nu(\mathcal{F}_{\mu\nu} + \{\mathcal{H}, \mathcal{K}_{\mu\nu}\} + i[\mathcal{K}_{\mu\rho}, \mathcal{K}_{\rho\nu}]) \\ & + i\Gamma_\mu\Gamma_5(\mathcal{D}_\mu\mathcal{H} + \{p_\nu - \mathcal{A}_\nu, \mathcal{K}_{\mu\nu}\}) - \frac{1}{2}\Gamma_{\mu\rho\sigma}\Gamma_5\mathcal{D}_\mu\mathcal{K}_{\rho\sigma} - \frac{1}{4}\Gamma_{\mu\nu\rho\sigma}\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma},\end{aligned}\quad (17)$$

with enlarged background fields defined by

$$\mathcal{A}_\mu = \begin{pmatrix} A_\mu^L & 0 \\ 0 & A_\mu^R \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & iH \\ -iH^\dagger & 0 \end{pmatrix}, \quad \mathcal{K}_{\mu\nu} = \begin{pmatrix} 0 & iK_{\mu\nu}^s \\ -iK_{\mu\nu}^{s\dagger} & 0 \end{pmatrix}. \quad (18)$$

$\Gamma_{A_1\dots A_k} \equiv \Gamma_{[A_1}\dots\Gamma_{A_k]}$ denotes the anti-symmetrized product of k Γ matrices, and the field-strength and the covariant derivative have been defined as

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu], \quad \mathcal{D}_\mu\chi = \partial_\mu\chi - i[\mathcal{A}_\mu, \chi]. \quad (19)$$

The $\hat{\Sigma}^2$ operator is seen to be manifestly gauge and chiral covariant. It also contains Γ matrices to even powers only, and is well suited for the worldline path integral representation.

2. Worldline Path Integral

With the use of the coherent state formalism [22, 23], one can perform the transition from Γ matrices to a path integral over Grassman fields ψ , with the correspondence $\Gamma_A\Gamma_B \rightarrow 2\psi_A\psi_B$ and $\Gamma_A\Gamma_B\Gamma_C\Gamma_D \rightarrow 4\psi_A\psi_B\psi_C\psi_D$, as long as A, B, C , and D are all different. The final form for the real part of the effective action is

$$W^+ = \frac{1}{8} \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \int_{AP} \mathcal{D}\psi \operatorname{tr} \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)}. \quad (20)$$

Here \mathcal{N} denotes a normalization constant coming from a momentum integration and AP stands for antiperiodic boundary conditions, which must be fulfilled by the Grassman variables $\psi(T) = -\psi(0)$. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(\tau) = & \frac{\dot{x}^2}{4} + \frac{1}{2}\psi_A\dot{\psi}_A - i\dot{x}_\mu\mathcal{A}_\mu + \mathcal{H}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}_{\mu\nu} + 2i\psi_\mu\psi_5(\mathcal{D}_\mu\mathcal{H} + i\dot{x}_\nu\mathcal{K}_{\mu\nu}) \\ & + i\psi_\mu\psi_\nu(\mathcal{F}_{\mu\nu} + \{\mathcal{H}, \mathcal{K}_{\mu\nu}\} + i[\mathcal{K}_{\mu\rho}, \mathcal{K}_{\rho\nu}]) \\ & - \psi_\mu\psi_\nu\psi_\rho(2\psi_5\mathcal{D}_\mu\mathcal{K}_{\mu\nu} + \psi_\sigma\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma}).\end{aligned}\quad (21)$$

The periodic boundary conditions for the field $x(\tau)$ suggest to separate the zero modes of the free field operator $\frac{d^2}{d\tau^2}$. The fields $x(\tau)$ are split into a constant part and a τ dependent

part according to $x(\tau) = x_0 + y(\tau)$, with $\partial_\tau x_0 = 0$ and $\int_0^T d\tau y(\tau) = 0$, and the measure in the integral is changed into $\mathcal{D}x = \mathcal{D}y d^D x_0$. The Green function is defined on a subspace orthogonal to the zero modes. The ψ_A fields contain no zero modes, so that the propagators for the $y(\tau)$ and $\psi_A(\tau)$ fields read

$$\begin{aligned}\langle y(\tau_1)y(\tau_2) \rangle &= \frac{(\tau_1 - \tau_2)^2}{T} - |\tau_1 - \tau_2|, \\ \langle \psi_A(\tau_1)\psi_B(\tau_2) \rangle &= \frac{1}{2}\delta_{AB}\text{sign}(\tau_1 - \tau_2).\end{aligned}\tag{22}$$

This formalism can then be used to determine the real part of the effective action as discussed in ref. [19].

B. Imaginary Part of the Effective Action

As in the case of the real part of the effective action, one requires a positive operator in order to use the Schwinger trick. Even though this is still possible for the imaginary part, gauge and chiral invariance cannot be manifestly conserved due to the chiral anomaly. For example, in ref. [19, 22] a parameter α is introduced, which breaks the chiral invariance, but leads to a positive operator. However the resulting expression is not appropriate for higher order calculations since the breaking of manifest chiral invariance leads to a large number of contributions in the perturbative expansion of the path integral.

The aim of the present work is to present a worldline representation of the effective current for which a manifestly chiral covariant expression exists. This current can then be integrated to obtain the effective action [1, 24, 25]. This integration rather proceeds by matching: First, a general effective action is proposed, which has the expected chiral and covariant properties. The functional variation of this action is then matched to the covariant current that is obtained using the worldline formalism. This method has the advantage that it is both gauge and chiral invariant at each stage of the calculation. The anomaly only leads to additional complications in the matching procedure of the lowest order contributions as will be discussed in detail in the next section.

Starting point of our analysis is the functional derivative of the imaginary part of the effective action in Eq. (6)

$$\delta W^- = \frac{1}{2}\delta \left(\log \text{Det } \mathcal{O} - \log \text{Det } \mathcal{O}^\dagger \right) = \frac{1}{2}\text{Tr} \left(\delta \mathcal{O} \frac{1}{\mathcal{O}} - \delta \mathcal{O}^\dagger \frac{1}{\mathcal{O}^\dagger} \right).\tag{23}$$

This expression can be rewritten in terms of a positive operator which can be used to employ the worldline representation in combination with the heat kernel formula. Incidentally, it can also be expressed in a manifestly chiral covariant form, what simplifies higher order calculations tremendously as compared to the formalism presented in ref. [19].

1. Construction of a Positive Operator for the Imaginary Part of the Effective Action

The expression in Eq. (23) can be transformed using the operator Σ defined in Eq. (8)

$$\delta W^- = \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & \delta \mathcal{O} \\ -\delta \mathcal{O}^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/\mathcal{O}^\dagger \\ 1/\mathcal{O} & 0 \end{pmatrix}, \quad (24)$$

which, with the introduction of a new matrix χ , can be rewritten as

$$\delta W^- = \frac{1}{2} \text{Tr} \chi \delta \Sigma \Sigma^{-1}, \quad (25)$$

with

$$\Sigma = \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O}^\dagger & 0 \end{pmatrix}, \quad \chi = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}. \quad (26)$$

To produce the positive definite operator Σ^2 in Eq. (25), we multiply and divide by Σ , using the cyclic property of the trace and the fact that Σ anticommutes with χ , to obtain

$$\begin{aligned} \delta W^- &= \frac{1}{4} \text{Tr} (\chi \delta \Sigma \Sigma + \Sigma \chi \delta \Sigma) \Sigma^{-2} \\ &= \frac{1}{4} \text{Tr} \chi [\delta \Sigma, \Sigma] \Sigma^{-2}. \end{aligned} \quad (27)$$

Since the last factor is a positive operator, it can be reexpressed as an integral, similar to the expression of the real part of the effective action in Eq. (17), namely

$$\delta W^- = \frac{1}{4} \text{Tr} \int_0^\infty dT \chi [\delta \Sigma, \Sigma] e^{-T \Sigma^2}. \quad (28)$$

As in the case for the real part, the chiral covariance can be made manifest by changing to an appropriate basis. With the help of the matrix M in Eq. (13), one obtains again

$$\tilde{\Sigma} = \gamma_\mu (p_\mu - \mathcal{A}_\mu) - \gamma_5 \mathcal{H} - \frac{i}{2} \gamma_\mu \gamma_\nu \gamma_5 \mathcal{K}_{\mu\nu}. \quad (29)$$

The additional factors $\chi [\delta \Sigma, \Sigma]$ read

$$M^{-1} \chi M = \tilde{\chi} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix} = \chi \gamma_5, \quad (30)$$

and for the case $\delta\tilde{\Sigma} = -\gamma_\mu\delta\mathcal{A}_\mu$

$$\begin{aligned} \left[\delta\tilde{\Sigma}, \tilde{\Sigma}\right] = & -\gamma_{\mu\nu}\{\delta\mathcal{A}_\mu, p_\nu - \mathcal{A}_\nu\} - i\mathcal{D}_\mu\delta\mathcal{A}_\mu - \gamma_5\gamma_\mu\{\delta\mathcal{A}_\mu, \mathcal{H}\} \\ & + i\gamma_5\gamma_\mu[\delta\mathcal{A}_\nu, \mathcal{K}_{\mu\nu}] - \frac{i}{2}\gamma_5\gamma_{\mu\lambda\sigma}\{\delta\mathcal{A}_\mu, \mathcal{K}_{\lambda\sigma}\}. \end{aligned} \quad (31)$$

To use the coherent state formalism, it is again necessary to enlarge the Clifford algebra and to replace the γ matrices by Γ matrices. However, taking into account the factor γ_5 in Eq. (30) the imaginary part of the effective action contains only odd combinations of γ matrices. Thus, the replacement

$$\gamma_A \rightarrow \Gamma_A = \gamma_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A \in [1 \dots 5] \quad (32)$$

has to be compensated by a factor

$$-\frac{i}{2}\Gamma_7\Gamma_6 = \mathbb{1}_4 \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \quad (33)$$

The overall factor Γ_7 changes the boundary condition of the fermionic sector from antiperiodic to periodic as explained in ref. [22]. This means that the fermionic sector contains zero modes, which have to be separated in the same way as was done for the bosonic sector.

Including the factor in Eq. (33) to compensate for the doubling of the Clifford space, one obtains

$$\delta W^- = \frac{i}{8}\text{Tr} \int_0^\infty dT \Gamma_7\Gamma_6 \chi w(T) e^{-T\hat{\Sigma}^2}, \quad (34)$$

where $\hat{\Sigma}^2$ is given in Eq. (17), and the insertion due to the commutator yields

$$\begin{aligned} w(T) = & -\frac{1}{2}\Gamma_5\Gamma_{\mu\nu}\{\delta\mathcal{A}_\mu, p_\nu - \mathcal{A}_\nu\} - i\Gamma_5\mathcal{D}_\mu\delta\mathcal{A}_\mu - \Gamma_\mu\{\delta\mathcal{A}_\mu, \mathcal{H}\} \\ & + i\Gamma_\mu[\delta\mathcal{A}_\nu, \mathcal{K}_{\mu\nu}] - \frac{i}{2}\Gamma_{\mu\lambda\sigma}\{\delta\mathcal{A}_\mu, \mathcal{K}_{\lambda\sigma}\}. \end{aligned} \quad (35)$$

To transform this expression into a worldline path integral, a similar procedure as for the real part of the effective action can be followed. Products of Γ matrices can be replaced by Grassman fields, however in this case the Jacobian of the transformation contains additional contributions from the zero modes

$$\begin{aligned} \mathcal{D}\theta\mathcal{D}\bar{\theta} & \equiv d\theta_3 d\theta_2 d\theta_1 d\bar{\theta}_1 d\bar{\theta}_2 d\bar{\theta}_3 \mathcal{D}\theta' \mathcal{D}\bar{\theta}' \\ & = \frac{1}{J} d\psi_1^0 d\psi_2^0 d\psi_3^0 d\psi_4^0 d\psi_5^0 d\psi_6^0 \mathcal{D}\psi'. \end{aligned} \quad (36)$$

The factor J only includes the Jacobian for the zero modes, while the Jacobian for the orthogonal modes is absorbed in the normalization of the correlation functions of the ψ'_A . J can be calculated from the definition of the Grassman fields ψ in the coherent state formalism [22] and yields in D dimension

$$J = \det \left(\frac{\partial \theta, \bar{\theta}}{\partial \psi} \right) = (-i)^{(D+2)/2}. \quad (37)$$

The final result can be expressed as

$$\delta W^- = \frac{1}{8} \text{tr} \int_0^\infty dT \mathcal{N} \int \mathcal{D}x \int_P \mathcal{D}\psi \chi w(T) \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)}. \quad (38)$$

The Lagrangian is of the same form as in the real part, Eq. (21),

$$\begin{aligned} \mathcal{L}(\tau) = & \frac{\dot{x}^2}{4} + \frac{1}{2} \psi_A \dot{\psi}_A - i \dot{x}_\mu \mathcal{A}_\mu + \mathcal{H}^2 - \frac{1}{2} \mathcal{K}_{\mu\nu} \mathcal{K}_{\mu\nu} + 2i \psi_\mu \psi_5 (\mathcal{D}_\mu \mathcal{H} + i \dot{x}_\nu \mathcal{K}_{\mu\nu}) \\ & + i \psi_\mu \psi_\nu (\mathcal{F}_{\mu\nu} + \{\mathcal{H}, \mathcal{K}_{\mu\nu}\}) - 2\psi_\mu \psi_\nu \psi_\rho \left(\psi_5 \mathcal{D}_\mu \mathcal{K}_{\mu\nu} + \frac{1}{2} \psi_\sigma \mathcal{K}_{\mu\nu} \mathcal{K}_{\rho\sigma} \right). \end{aligned} \quad (39)$$

and the trivial integration over ψ_6 can be carried out, so that the insertion yields

$$\begin{aligned} w(T) = & -4i \psi_5 \psi_\mu \psi_\nu \delta \mathcal{A}_\mu \dot{x}_\nu - 2i \psi_5 \mathcal{D}_\mu \delta \mathcal{A}_\mu - 2\psi_\mu \{\delta \mathcal{A}_\mu, \mathcal{H}\} \\ & + 2i \psi_\mu [\delta \mathcal{A}_\nu, \mathcal{K}_{\mu\nu}] - 2i \psi_\mu \psi_\lambda \psi_\sigma \{\delta \mathcal{A}_\mu, \mathcal{K}_{\lambda\sigma}\}. \end{aligned} \quad (40)$$

The normalization \mathcal{N} coming from the momentum integration, satisfies

$$\mathcal{N} \int \mathcal{D}x e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} = (4\pi T)^{-D/2} \int d^D x. \quad (41)$$

The Green function for the bosonic field x is the same as for the real part of the effective action, Eq. (22), while the Green function of the Grassman fields ψ_A differs due to the presence of the zero modes. The fermionic fields are split according to $\psi_A(\tau) = \psi_A^0 + \psi'_A(\tau)$, with $\partial_\tau \psi_A^0 = 0$ and $\int_0^T d\tau \psi'_A(\tau) = 0$ and the measure turns into $\mathcal{D}\psi = d\psi_1 d\psi_2 d\psi_3 d\psi_4 d\psi_5 \mathcal{D}\psi'$. The Green function for the ψ'_A fields, defined on a space orthogonal to the zero modes, reads

$$\left\langle \psi'_A(\tau_1) \psi'_B(\tau_2) \right\rangle = \delta_{AB} \left(\frac{1}{2} \text{sign}(\tau_1 - \tau_2) - \frac{(\tau_1 - \tau_2)}{T} \right). \quad (42)$$

These results can be easily generalized to different dimensions. In two dimension, one obtains an additional overall factor $-i$ from the Jacobian of the zero modes and the fermionic measure reads $\mathcal{D}\psi = d\psi_1 d\psi_2 d\psi_5 \mathcal{D}\psi'$.

2. The Effective Density

The effective density is obtained by varying with respect to the \mathcal{H} field, so that $\delta\tilde{\Sigma} = -\gamma_5\delta\mathcal{H}$. In comparison to the worldline representation of the covariant current only the insertion changes into

$$\left[\delta\tilde{\Sigma}, \tilde{\Sigma}\right] = -\gamma_5\gamma_\mu \{\delta\mathcal{H}, p_\mu - \mathcal{A}_\mu\} + [\delta\mathcal{H}, \mathcal{H}] + \frac{i}{2}\gamma_\mu\gamma_\nu [\delta\mathcal{H}, \mathcal{K}_{\mu\nu}]. \quad (43)$$

The corresponding insertion $w(T)$ in the path integral reads then

$$w(T) = -2i\psi_\mu\dot{x}_\mu\delta\mathcal{H} + 2\psi_5[\delta\mathcal{H}, \mathcal{H}] + 2i\psi_\mu\psi_\nu[\delta\mathcal{H}, \mathcal{K}_{\mu\nu}]. \quad (44)$$

Since $\delta\mathcal{A}$ carries an index, the effective current is of one order lower than the effective density and usually results in less terms to calculate. The advantage of the effective density lies in the matching process, since the factors in the effective density consist of the same type as found in the effective action. They both combine the same type of object, \mathcal{DH} and \mathcal{F} , to the same kind of order, while the effective current combines the terms to a lower order. Besides, there is no distinction between a consistent effective density and a covariant effective density, as there is for the effective current, as will be explained in the next section.

3. Distinction between the Consistent and the Covariant Current

With Eq. (38) an expression for the covariant current which is chiral and gauge covariant was derived. This current cannot be the variation of the effective action, since the effective action contains the chiral anomaly, and in fact the covariant current is not a variation of any action. The reason for this is that performing the variation does not commute with the regularization procedure we used, namely the Schwinger trick. On the other hand, knowing the chiral anomaly, one can reproduce the so-called consistent current that denotes the true variation of the effective action.

To explain the relation between the two currents, we define a general variation

$$\delta_Y = \int dx Y_\mu^a(x) \frac{\delta}{\delta\mathcal{A}_\mu(x)}, \quad (45)$$

so that a gauge variation δ_ξ is given by

$$\delta_\xi = \int dx (\mathcal{D}_\mu\xi)(x) \frac{\delta}{\delta\mathcal{A}_\mu(x)}. \quad (46)$$

Two subsequent variations have then the commutator $[\delta_Y, \delta_\xi] = \delta_{[Y, \xi]}$ and in order to find the transformation properties of the consistent current, one can apply this commutator to the effective action

$$[\delta_Y, \delta_\xi]W^-[\mathcal{A}_\mu] = \delta_{[Y, \xi]}W^-[\mathcal{A}_\mu]. \quad (47)$$

Using the anomalous Ward identity [26]

$$\delta_\xi W^-[\mathcal{A}_\mu] = \int dx \xi(x) G[\mathcal{A}_\mu](x), \quad (48)$$

with $G[\mathcal{A}_\mu]$ denoting the consistent anomaly, one can evaluate both sides of Eq. (47) to obtain

$$\begin{aligned} \int dx [Y_\mu, \xi](x) \frac{\delta}{\delta \mathcal{A}_\mu(x)} W^-[\mathcal{A}_\mu] &= \delta_Y \int dx \xi(x) G[\mathcal{A}_\mu](x) \\ &\quad - \delta_\xi \int dx Y_\mu(x) \frac{\delta}{\delta \mathcal{A}_\mu(x)} W^-[\mathcal{A}_\mu]. \end{aligned} \quad (49)$$

Defining the consistent current as the variation of the effective action

$$\langle j^\mu(x) \rangle = \frac{\delta}{\delta \mathcal{A}_\mu(x)} W^-[\mathcal{A}_\mu], \quad (50)$$

one finds

$$\int dx Y_\mu(x) \delta_\xi \langle j^\mu(x) \rangle = \int dx Y_\mu [\langle j^\mu(x) \rangle, \xi](x) + \int dx \xi(x) \delta_Y G[\mathcal{A}_\mu](x). \quad (51)$$

Since Y was a general variation this leads to

$$\delta_\xi \langle j^\mu(x) \rangle = [\langle j^\mu(x) \rangle, \xi] + \int dy \xi^b(y) \frac{\delta}{\delta \mathcal{A}_\mu(x)} G[\mathcal{A}_\mu](y). \quad (52)$$

This shows that only if the anomaly vanishes, the current transforms covariantly. This relation can be used to determine the connection between the consistent current, i.e. the true variation of the action, and the covariant current. The latter is obtained by adding an object $P^\mu[\mathcal{A}_\mu]$, called the Bardeen-Zumino polynomial [27], to the consistent current so that the sum transforms covariantly

$$\langle \bar{j}^\mu \rangle = \langle j^\mu \rangle + \langle P^\mu \rangle. \quad (53)$$

This implies the following gauge transformation property for the BZ polynomial

$$\delta_\xi P^\mu[\mathcal{A}_\mu](x) = [P^\mu[\mathcal{A}_\mu], \xi](x) - \int dy \xi(y) \frac{\delta}{\delta \mathcal{A}_\mu(x)} G[\mathcal{A}_\mu](y). \quad (54)$$

It is not obvious that such an object exists, but using

$$P^\mu[\mathcal{A}_\mu] = \frac{1}{48\pi^2} \epsilon^{\mu\nu\lambda\sigma} \text{tr} \chi (\mathcal{A}_\nu \mathcal{F}_{\lambda\sigma} + \mathcal{F}_{\lambda\sigma} \mathcal{A}_\nu + i \mathcal{A}_\nu \mathcal{A}_\lambda \mathcal{A}_\sigma), \quad (55)$$

and the consistent anomaly [26]

$$G[\mathcal{A}_\mu] = \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda\sigma} \text{tr} \chi \partial_\mu \left(\mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma - \frac{i}{2} \mathcal{A}_\nu \mathcal{A}_\lambda \mathcal{A}_\sigma \right), \quad (56)$$

it can be shown that the definition of P^μ in Eq. (55) provides a unique polynomial in \mathcal{A}_μ that satisfies Eq. (54). The corresponding functions in two dimensions are given by

$$P^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu} \text{tr} \mathcal{A}_\nu, \quad G[\mathcal{A}_\mu] = \frac{1}{4\pi} \epsilon^{\mu\nu} \text{tr} \chi \partial_\mu \mathcal{A}_\nu. \quad (57)$$

As stated above, the path integral in Eq. (38) constitutes a worldline representation of the covariant current. To obtain the imaginary part of the effective action from the covariant current one can use the following ansatz

$$W^- = \Gamma_{gWZW} + W_c^-. \quad (58)$$

Here, Γ_{gWZW} is an extended gauged Wess-Zumino-Witten action [24, 28, 29], which is chosen to reproduce the correct chiral anomaly, and W_c^- denotes a chiral invariant part. The variation of the functional Γ_{gWZW} , consists of a part that saturates the anomaly, namely the BZ polynomial, and a covariant remainder which has to be added to the variation of W_c^- to yield the covariant current.

4. The Wess-Zumino-Witten action

When the effective action is separated into two parts, it is required by the non-covariant part that it reproduces the anomaly. It is well known that the WZW action has this property.

The ungauged WZW action in four dimension is e.g. of the form

$$\Gamma(\mathbf{U}) = \frac{i}{48\pi^2} \int_Q d^5x \epsilon^{abcde} \text{tr} \left[\frac{1}{5} \mathbf{U}^{-1} \partial_a \mathbf{U} \mathbf{U}^{-1} \partial_b \mathbf{U} \mathbf{U}^{-1} \partial_c \mathbf{U} \mathbf{U}^{-1} \partial_d \mathbf{U} \mathbf{U}^{-1} \partial_e \mathbf{U} \right], \quad (59)$$

where Q is a five-dimensional space with boundary ∂Q equal to the R^4 flat Euclidean space. The matrix \mathbf{U} is a unitary matrix, and is usually related to the case where the mass can be expressed as a constant times that unitary matrix. We are interested in the more general case when the mass matrix is not of this form which is called extended WZW

action. In addition, the presence of the background gauge fields makes a gauging of the action mandatory. The gauged extended WZW action can be generally expressed as the integral in five dimensions [24]. Unlike the action itself, the resulting current turns out to be a total derivative in five dimensions, such that it can be represented by an integral over the physical four-dimensional space

$$\begin{aligned} \delta\Gamma_{gWZW} = & \frac{1}{96\pi^2} \int d^4x \epsilon^{\mu\nu\lambda\sigma} \text{tr} \chi \left[\delta\mathcal{A}_\mu \left(-\mathcal{H}^{-1} \mathcal{D}_\nu \mathcal{H} \mathcal{H}^{-1} \mathcal{D}_\lambda \mathcal{H} \mathcal{H}^{-1} \mathcal{D}_\sigma \mathcal{H} \right. \right. \\ & + \mathcal{D}_\nu \mathcal{H} \mathcal{H}^{-1} \mathcal{D}_\lambda \mathcal{H} \mathcal{H}^{-1} \mathcal{D}_\sigma \mathcal{H} \mathcal{H}^{-1} - i \{ \mathcal{H}^{-1} \mathcal{D}_\nu \mathcal{H} - \mathcal{D}_\nu \mathcal{H} \mathcal{H}^{-1}, \mathcal{F}_{\lambda\sigma} \} \\ & + \frac{i}{2} \mathcal{H} \{ \mathcal{H}^{-1} \mathcal{D}_\nu \mathcal{H}, \mathcal{F}_{\lambda\sigma} \} \mathcal{H}^{-1} - \frac{i}{2} \mathcal{H}^{-1} \{ \mathcal{D}_\nu \mathcal{H} \mathcal{H}^{-1}, \mathcal{F}_{\lambda\sigma} \} \mathcal{H} \\ & \left. \left. - 2 \{ \mathcal{A}_\nu, \mathcal{F}_{\lambda\sigma} \} - 2i \mathcal{A}_\nu \mathcal{A}_\lambda \mathcal{A}_\sigma \right) \right], \end{aligned} \quad (60)$$

or in two dimensions

$$\delta\Gamma_{gWZW} = \frac{1}{8\pi} \int d^2x \epsilon^{\mu\nu} \text{tr} \chi \left[\delta\mathcal{A}_\mu \left(-i \mathcal{H}^{-1} \mathcal{D}_\nu \mathcal{H} + i \mathcal{D}_\nu \mathcal{H} \mathcal{H}^{-1} - 2\mathcal{A}_\nu \right) \right]. \quad (61)$$

Notice that in both cases the last term in the current denotes the BZ polynomial. The remaining chiral covariant terms have to be subtracted from the covariant current before it is matched to the effective action according to the ansatz made in Eq. (58).

III. LOWEST ORDER EFFECTIVE ACTION

A. Effective covariant current

In order to reproduce the results from ref. [1], we neglect in this section the antisymmetric field $K_{\mu\nu}$. The fields \mathcal{A} and \mathcal{H} are matrices of some internal group, and we only assume that $\mathcal{H}(x_0)$ is nowhere singular. With this in mind, we restate our result Eq. (38) from the last section in D dimensions

$$\delta W^- = -\frac{i^{D/2}}{8} \text{tr} \int_0^\infty dT \mathcal{N} \int \mathcal{D}x \int_P \mathcal{D}\psi \chi w(T) \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)}, \quad (62)$$

with

$$\begin{aligned} \mathcal{L}(\tau) &= \frac{\dot{x}^2}{4} + \frac{1}{2} \psi_A \dot{\psi}_A - i \dot{x}_\mu \mathcal{A}_\mu + \mathcal{H}^2 + 2i \psi_\mu \psi_5 \mathcal{D}_\mu \mathcal{H} + i \psi_\mu \psi_\nu \mathcal{F}_{\mu\nu}, \\ w(T) &= -4i \psi_5 \psi_\mu \psi_\nu \delta \mathcal{A}_\mu \dot{x}_\nu - 2i \psi_5 \mathcal{D}_\mu \delta \mathcal{A}_\mu - 2\psi_\mu \{ \delta \mathcal{A}_\mu, \mathcal{H} \}. \end{aligned} \quad (63)$$

Next, the derivative expansion of the heat kernel is used. In the derivative expansion terms are classified by the number of covariant indices that they carry, so that $\mathcal{D}_\mu \mathcal{H}$ is of first order,

while $\mathcal{F}_{\mu\nu}$ is of second order. The worldline formalism is well suited for this expansion, and there are two major advantages compared to the more traditional methods used e.g. in ref. [1]. First, the tedious manipulations using the γ algebra are avoided. Secondly, the momentum integration is omitted and replaced by the rather trivial integration in τ space.

The coordinate is split as $x(\tau) = x_0 + y(\tau)$, and we work in the Fock-Schwinger gauge [30], in which $\mathcal{A}(x) \cdot y = 0$. In this gauge, expressions remain gauge covariant and the field \mathcal{A} can be expressed in terms of the field strength tensor $\mathcal{F}_{\mu\nu}$ by

$$\mathcal{A}_\mu(x) = \int_0^1 d\alpha \alpha \mathcal{F}_{\rho\mu}(x_0 + \alpha y) y_\rho. \quad (64)$$

All background fields can then be expanded around the point x_0 in terms of covariant derivatives

$$X(x_0 + y(\tau)) = \exp(y(\tau) \cdot \mathcal{D}_{x_0}) X(x_0), \quad (65)$$

where \mathcal{D}_{x_0} refers to the covariant derivative in Eq. (19) with respect to x_0 . With the expansion of the field strength tensor in terms of covariant derivatives and Eq. (64), one can rewrite the field \mathcal{A} as

$$\mathcal{A}_\mu(x) = \frac{1}{2} y_\rho \mathcal{F}_{\rho\mu}(x_0) + \frac{1}{3} y_\alpha y_\rho \mathcal{D}_\alpha \mathcal{F}_{\rho\mu}(x_0) + \frac{1}{4 \cdot 2!} y_\alpha y_\beta y_\rho \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{F}_{\rho\mu}(x_0) + \dots \quad (66)$$

Since we will not carry out the integration with respect to x_0 we use the following notation in D dimensions

$$\langle X \rangle_D = - \left(\frac{i}{4\pi} \right)^{D/2} \text{tr} \chi \int d^D x_0 X. \quad (67)$$

It is important to remember that χ and \mathcal{H} anticommute; hence, when the cyclic property of the trace is used, a minus sign is generated, for example

$$\begin{aligned} \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{H} \mathcal{F}_{\mu\nu} \mathcal{H}^3 \mathcal{F}_{\lambda\sigma} \rangle &= - \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{F}_{\mu\nu} \mathcal{H}^3 \mathcal{F}_{\lambda\sigma} \mathcal{H} \rangle = - \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{H}^3 \mathcal{F}_{\lambda\sigma} \mathcal{H} \mathcal{F}_{\mu\nu} \rangle \\ &= - \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{H}^3 \mathcal{F}_{\mu\nu} \mathcal{H} \mathcal{F}_{\lambda\sigma} \rangle. \end{aligned} \quad (68)$$

After expanding the mass field $\mathcal{H}(x)^2 = \mathcal{H}^2(x_0) + y_\mu \mathcal{D}_\mu \mathcal{H}^2(x_0) + \dots$, the field $\mathcal{H}(x_0)$ is treated non-perturbatively. Since all the fields can be matrices of some internal space the resulting expressions normally cannot be expressed in closed form. For this case we use the labeled operator notation laid down in ref. [2, 31]. The notation works as follows: In an expression $f(A_1, B_2, \dots)XY\dots$, the labels of the operators A, B, \dots denote the position of that operator with respect to the remaining operators $XY\dots$. For instance, for the function

$f(A, B) = \alpha(A)\beta(B)$, the expression $f(A_1, B_2)XY$ represents $\alpha(A)X\beta(B)Y$. In the case at hand, the operator appearing in the functions is always $m := \mathcal{H}(x_0)$, such that general functions f can be easily interpreted in the basis where m is diagonal. Using this notation, Eq. (68) can be recast as

$$\begin{aligned} \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{H} \mathcal{F}_{\mu\nu} \mathcal{H}^3 \mathcal{F}_{\lambda\sigma} \rangle &= \langle \epsilon^{\mu\nu\lambda\sigma} m_1 m_2^3 \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\sigma} \rangle = - \langle \epsilon^{\mu\nu\lambda\sigma} m_3 m_2^3 \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\sigma} \rangle \\ &= - \langle \epsilon^{\mu\nu\lambda\sigma} m_2 m_1^3 \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\sigma} \rangle = - \langle \epsilon^{\mu\nu\lambda\sigma} \mathcal{H}^3 \mathcal{F}_{\mu\nu} \mathcal{H} \mathcal{F}_{\lambda\sigma} \rangle. \end{aligned} \quad (69)$$

This notation can also be used to simplify the matrix valued derivative. Using the definition

$$(\nabla f)(m_1, m_2) := \frac{f(m_1) - f(m_2)}{m_1 - m_2}, \quad (70)$$

it is possible to prove that

$$\mathcal{D}_\mu f(m) = (\nabla f)(m_1, m_2) \mathcal{D}_\mu \mathcal{H}. \quad (71)$$

For example, in the polynomial case $f(m) = m^3$ one obtains

$$\begin{aligned} \mathcal{D}_\mu f(m) &= \mathcal{D}_\mu (\mathcal{H}^3) = \mathcal{D}_\mu \mathcal{H} \mathcal{H}^2 + \mathcal{H} \mathcal{D}_\mu \mathcal{H} \mathcal{H} + \mathcal{H}^2 \mathcal{D}_\mu \mathcal{H} \\ &= (m_2^2 + m_1 m_2 + m_1^2) \mathcal{D}_\mu \mathcal{H} = \frac{m_1^3 - m_2^3}{m_1 - m_2} \mathcal{D}_\mu \mathcal{H} \\ &= (\nabla f)(m_1, m_2) \mathcal{D}_\mu \mathcal{H}. \end{aligned} \quad (72)$$

As mentioned earlier, non-polynomial expressions are hereby interpreted in a basis where m is diagonal, so that for $m = \text{diag}(d_1, \dots, d_n)$

$$\frac{f(m_1) - f(m_2)}{m_1 - m_2} X = \frac{f(d_i) - f(d_j)}{d_i - d_j} X_{ij}. \quad (73)$$

More general, this suggests the following definition for the case with several variables:

$$\nabla_k f(m_1, \dots, m_n) = \frac{f(m_1, \dots, \hat{m}_{k+1}, \dots, m_n) - f(m_1, \dots, \hat{m}_k, \dots, m_n)}{m_k - m_{k+1}}, \quad (74)$$

where \hat{m}_k indicates that the corresponding argument is left out.

If all arguments of the functions are of the same type one can further simplify the notation and use subscripts to refer to the argument of the function, e.g. $f(m_1, m_2) =: f_{12}$ and we employ this notation in the following. Additionally, negative arguments will be denoted by underlining the corresponding index, $f(-m_1, m_2) =: f_{\underline{1}2}$. More applications of the labeled operator notation can be found in refs. [1, 2].

The path ordering in Eq. (62) is defined by

$$\mathcal{P} \prod_{i=1}^N \int_0^T d\tau_i \equiv N! \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-1}} d\tau_N = N! \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \prod_{i=1}^{N-1} \theta(\tau_i - \tau_{i+1}). \quad (75)$$

Separating the Lagrangian Eq. (63) into $\mathcal{L}(\tau) = \mathcal{L}_0(\tau) + \mathcal{H}^2(x_0) + \mathcal{L}_1(\tau)$, with

$$\begin{aligned} \mathcal{L}_0(\tau) &= \frac{\dot{x}^2}{4} + \frac{1}{2} \psi_A \dot{\psi}_A, \\ \mathcal{L}_1(\tau) &= -i \dot{x}_\mu \mathcal{A}_\mu(x) + 2i \psi_\mu \psi_5 \mathcal{D}_\mu \mathcal{H}(x) + i \psi_\mu \psi_\nu \mathcal{F}_{\mu\nu}(x) + y_\mu \mathcal{D}_\mu \mathcal{H}^2(x_0) + \dots, \end{aligned} \quad (76)$$

the terms of the expansion of $\mathcal{H}^2(x)$, except the leading term $\mathcal{H}^2(x_0)$, are attributed to $\mathcal{L}_1(\tau)$, and treated perturbatively. Notice that \mathcal{L}_0 commutes with the rest of the Lagrangian, so that the expansion of the path ordered exponential in Eq. (62) takes the form

$$\begin{aligned} \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)} &= e^{-\int_0^T d\tau \mathcal{L}_0(\tau)} \left(e^{-T \mathcal{H}^2(x_0)} + \int_0^T d\tau_1 e^{-(T-\tau_1) \mathcal{H}^2(x_0)} (-\mathcal{L}_1(\tau_1)) e^{-\tau_1 \mathcal{H}^2(x_0)} \right. \\ &\quad \left. + \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-(T-\tau_1) \mathcal{H}^2(x_0)} (-\mathcal{L}_1(\tau_1)) e^{-(\tau_1-\tau_2) \mathcal{H}^2(x_0)} (-\mathcal{L}_1(\tau_2)) e^{-\tau_2 \mathcal{H}^2(x_0)} + \dots \right). \end{aligned} \quad (77)$$

When performing the ψ integrals, the zero modes have to be saturated and at least a factor $\psi_1^0 \dots \psi_D^0 \psi_5^0$ is required from the Grassman fields in order to contribute. The first term in Eq. (77) lacks the appropriate ψ factor except in two dimensions, where the first term of the insertion Eq. (63) already has the appropriate factor. However it contains a factor \dot{x}_μ which must be contracted with a similar factor to form a Green function, hence it does not contribute and can be left out. The rest of Eq. (77) can be simplified using the labeled operator notation. Using the expression m_n^2 to denote $\mathcal{H}^2(x_0)$ in n th position, one obtains

$$\begin{aligned} \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)} &= e^{-\int_0^T d\tau \mathcal{L}_0(\tau)} \left(- \int_0^T d\tau_1 e^{-T m_1^2 - \tau_1 (m_2^2 - m_1^2)} \mathcal{L}_1(\tau_1) \right. \\ &\quad \left. + \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-T m_1^2 - \tau_1 (m_2^2 - m_1^2) - \tau_2 (m_3^2 - m_2^2)} \mathcal{L}_1(\tau_1) \mathcal{L}_1(\tau_2) + \dots \right). \end{aligned} \quad (78)$$

The evaluation of the worldline path integral can be summarized as follows: First, all fields in Eq. (78) and the insertion are expanded around x_0 . Next, the functional integration over the y fields is carried out, generating bosonic Green functions. Then, the ψ integrations are performed saturating the zero modes and generating fermionic Green functions. Finally, the T and τ integrations are performed.

Before presenting the actual calculation, we comment on the behavior of the effective action under complex conjugation. As noted earlier, in any contribution to the imaginary

part of the action the field ψ_5 appears an odd number of times. If one attributes a factor i to the operators \mathcal{F} and $\delta\mathcal{A}$ one observes that the remaining expressions in the current in Eq. (63) are real. Accordingly, all expressions in W^- are real as long as a factor i is attributed to the operator \mathcal{F} . In addition, notice that the effective action has to be an even function in the masses due to chiral invariance.

In order to showcase the method, we present the lowest order calculation in two dimensions. The lowest order contribution coming from the first term in the insertion is given by

$$-4i\psi_5\psi_\mu\psi_\nu\dot{y}_\nu(T)\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}y_\alpha(\tau)\mathcal{D}_\alpha\mathcal{H}^2\delta\mathcal{A}_\mu(T). \quad (79)$$

Performing the y and ψ integrals one obtains

$$\begin{aligned} &\frac{i}{2}\left\langle\epsilon^{\mu\nu}(m_1+m_2)\int_0^\infty\frac{dT}{T}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}\dot{y}_B(T,\tau_1)\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\right\rangle \\ &= \frac{i}{2}\left\langle\epsilon^{\mu\nu}J_{12}^2(m_1+m_2)\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\right\rangle. \end{aligned} \quad (80)$$

The second term of the insertion does not contribute at lowest order since it is already of second order in derivatives but lacks the appropriate fermionic factor to saturate the zero modes. The third term of the insertion leads only to one contribution of the form

$$-2\psi_\mu\{\delta\mathcal{A}_\mu,\mathcal{H}\}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}(-2i\psi_\nu\psi_5\mathcal{D}_\nu\mathcal{H}). \quad (81)$$

yielding

$$\begin{aligned} &-\frac{i}{2}\left\langle\epsilon^{\mu\nu}(m_1-m_2)\int_0^\infty\frac{dT}{T}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\right\rangle \\ &= -\frac{i}{2}\left\langle\epsilon^{\mu\nu}J_{12}^1(m_1-m_2)\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\right\rangle. \end{aligned} \quad (82)$$

The factor (m_1-m_2) results from the anticommutator in Eq. (81), and the sign change in the cyclic property of the trace as explained in Eq. (68). The integrals J are given in Appendix A. The total current is hence given by

$$\delta W^- = -i\left\langle\epsilon^{\mu\nu}A_{12}^1\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\right\rangle, \quad (83)$$

$$A_{12}^1 = \frac{1}{m_1-m_2} - \frac{m_1m_2\log(m_1^2/m_2^2)}{(m_1-m_2)(m_1^2-m_2^2)}, \quad (84)$$

where the function A_{12}^1 has been defined. This agrees with the results obtained in ref. [1].

B. Effective Density

The effective density can be obtained similar as the covariant current, utilizing the insertion in Eq. (44). Neglecting the antisymmetric tensor \mathcal{K} , the insertion is

$$w(T) = -2i \psi_\mu \dot{x}_\mu \delta \mathcal{H} + 2\psi_5 [\delta \mathcal{H}, \mathcal{H}]. \quad (85)$$

The contributions to the effective density are

$$\begin{aligned} \delta W^- &= \left\langle \epsilon^{\mu\nu} \left(\frac{i}{4} (J_{12}^1(m_1 + m_2) + J_{12}^2(m_1 - m_2)) \mathcal{F}_{\mu\nu} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} (J_{123}^5(m_1 + m_3) + J_{123}^6(m_1 + m_2) - J_{123}^7(m_2 + m_3)) \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \right) \delta \mathcal{H} \right\rangle \\ &= \left\langle \epsilon^{\mu\nu} \left(-\frac{i}{2} B_{12}^1 \mathcal{F}_{\mu\nu} + B_{123}^2 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \right) \delta \mathcal{H} \right\rangle. \end{aligned} \quad (86)$$

where the functions B_{12} and B_{123} are given by

$$B_{12}^1 = -\frac{1}{m_1 + m_2} - \frac{m_1 m_2}{(m_1 + m_2)(m_1^2 - m_2^2)} \log \left(\frac{m_1^2}{m_2^2} \right), \quad (87)$$

$$B_{123}^2 = B_{123}^R + B_{123}^L \log(m_1^2) + B_{231}^L \log(m_2^2) + B_{312}^L \log(m_3^2), \quad (88)$$

with

$$B_{123}^R = \frac{1}{(m_1 - m_2)(m_2 - m_3)(m_1 + m_3)}, \quad (89)$$

$$B_{123}^L = \frac{(m_1^3 + m_1 m_2 m_3)}{(m_1 - m_2)(m_1 + m_3)(m_1^2 - m_2^2)(m_1^2 + m_3^2)}, \quad (90)$$

in accordance with ref. [1].

C. Effective Action

We proceed and briefly present the derivation of the imaginary part of the effective action following ref. [1]. Using the ansatz in Eq. (58), the most general functional for W_c^- consistent with chiral and gauge invariance in two dimensions reads

$$W_c^- = \langle \epsilon^{\mu\nu} N_{12} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \rangle. \quad (91)$$

An additional term proportional to \mathcal{F} could be added but it can be removed by partial integration. Notice that N_{12} is a real function according to the comments made in the last section.

The function N_{12} has some nontrivial restrictions. First of all, the function N_{12} is even in m such that

$$N(-m_1, -m_2) := N_{\underline{12}} = N_{12}. \quad (92)$$

Because of the cyclic property of the trace one obtains

$$N_{12} = N_{32} = N_{21} = N_{\underline{21}}, \quad (93)$$

and due to the Hermiticity of W^-

$$N_{12} = -N_{32} = -N_{\underline{12}} = -N_{21}. \quad (94)$$

Varying $W_c^-[\mathcal{A}, \mathcal{H}]$ with respect to \mathcal{A} , one obtains

$$\delta W_c^- = -i \langle \epsilon^{\mu\nu} (-2(m_1 + m_2)N_{12}) \mathcal{D}_\mu \mathcal{H} \delta \mathcal{A}_\nu \rangle. \quad (95)$$

Comparing this to Eq. (83) and adding the covariant contribution in Eq. (61) coming from Γ_{gWZW} one has

$$\frac{1}{m_1 - m_2} - \frac{m_1 m_2 \log(m_1^2/m_2^2)}{(m_1 - m_2)(m_1^2 - m_2^2)} = \frac{1}{2m_1} - \frac{1}{2m_2} - 2(m_1 + m_2)N_{12}, \quad (96)$$

which finally leads to

$$N_{12} = \frac{1}{2} \frac{m_1 m_2}{m_1^2 - m_2^2} \left(\frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2} - \frac{1}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right). \quad (97)$$

At higher order, the matching of the effective potential to the current potentially becomes more intricate. On the other hand, the anomaly only contributes to the leading order, such that the knowledge of the covariant current (that in higher order coincides with the consistent current) suffices to determine the effective action.

D. Four Dimensions

For completeness, we also present the results for the effective action and the effective current in four dimensions. The matching procedure proceeds the same way as in ref. [1], and we do not repeat it here.

The effective current in four dimensions consists out of three terms and reads

$$\delta W_{d=4}^- = -i \left\langle \epsilon^{\mu\nu\lambda\sigma} \left(-\frac{i}{2} A_{123}^2 \mathcal{F}_{\nu\lambda} \mathcal{D}_\mu \mathcal{H} - \frac{i}{2} A_{123}^3 \mathcal{D}_\mu \mathcal{H} \mathcal{F}_{\nu\lambda} - A_{1234}^4 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \right) \delta \mathcal{A}_\sigma \right\rangle, \quad (98)$$

while the effective density can be written as

$$\begin{aligned} \delta W_{d=4}^- = & \left\langle \epsilon^{\mu\nu\lambda\sigma} \left(\frac{1}{4} B_{123}^3 \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\sigma} + \frac{i}{2} B_{1234}^4 \mathcal{F}_{\lambda\sigma} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \right. \right. \\ & + \frac{i}{2} B_{1234}^5 \mathcal{D}_\mu \mathcal{H} \mathcal{F}_{\lambda\sigma} \mathcal{D}_\nu \mathcal{H} + \frac{i}{2} B_{1234}^6 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{F}_{\lambda\sigma} \\ & \left. \left. - B_{12345}^7 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \mathcal{D}_\sigma \mathcal{H} \right) \delta \mathcal{H} \right\rangle. \end{aligned} \quad (99)$$

The functions A_{123}^2 , A_{123}^3 , A_{1234}^4 , B_{123}^3 , B_{1234}^4 , B_{1234}^5 , B_{1234}^6 , and B_{12345}^7 are given in Appendix B.

IV. NEXT TO LEADING ORDER EFFECTIVE ACTION IN TWO DIMENSIONS

In this section we present as a novel result the imaginary part of the effective action in next to leading order and two dimensions. Even though the results are rather lengthy, the evaluation of the worldline path integral involves only very basic integrals such that it can be easily implemented using a computer algebra system.

In two dimensions and in next to leading order, the imaginary part of the effective action takes the form

$$\begin{aligned} W^c = & \left\langle \epsilon^{\mu\nu} \left(Q_{12} \mathcal{D}_\mu \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\alpha \mathcal{D}_\nu \mathcal{H} + \frac{i}{2} P_{12} \mathcal{F}_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}_\alpha \mathcal{H} \right. \right. \\ & + \tilde{R}_{123} \mathcal{D}_\alpha \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} + \frac{i}{2} \hat{R}_{123} \mathcal{F}_{\mu\nu} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \\ & \left. \left. + M_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \right) \right\rangle. \end{aligned} \quad (100)$$

At next to leading order the action is chiral invariant and the effective action can hence be immediately obtained by matching with the covariant current that in this order coincides with the consistent current. These functions must have the following properties

$$P_{12} = -P_{\underline{12}} = P_{21}, \quad Q_{12} = Q_{\underline{12}} = -Q_{21}, \quad (101)$$

$$\tilde{R}_{123} = -\tilde{R}_{\underline{123}} = \tilde{R}_{\underline{213}}, \quad \hat{R}_{123} = \hat{R}_{\underline{123}} = -\hat{R}_{\underline{213}}, \quad (102)$$

$$M_{1234} = M_{\underline{1234}} = -M_{\underline{3412}} = M_{4321}. \quad (103)$$

We have chosen a rather general imaginary effective action at the required order which preserves gauge and chiral invariance, but we have included a larger number of terms than necessary to perform the matching process with the effective current. In fact, the matching

process could be done with solely the functions Q_{12} , \tilde{R}_{123} , \hat{R}_{123} , and M_{1234} . Instead, we have decided to include the additional term P_{12} , in order to have the option of simplifying the action by a judicious choice of this extra function. For example, the extra function can be used to ensure that all functions remain finite at the coincidence limit, as will be explained later on.

The calculation from the worldline formalism leads to the following contributions to the covariant current

$$\begin{aligned} \delta W^c = & -i \epsilon^{\mu\nu} \langle I_{12}^1 \mathcal{D}_\mu \mathcal{D}_\alpha \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu + i I_{12}^2 \mathcal{D}_\alpha \mathcal{F}_{\mu\alpha} \delta \mathcal{A}_\nu + I_{123}^3 \mathcal{D}_\alpha \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\mu \mathcal{H} \delta \mathcal{A}_\nu \\ & + I_{123}^4 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\alpha \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu + I_{123}^5 \mathcal{D}_\mu \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu + I_{123}^6 \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\mu \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu \\ & + i I_{123}^7 \mathcal{F}_{\mu\alpha} \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu + i I_{123}^8 \mathcal{D}_\alpha \mathcal{H} \mathcal{F}_{\mu\alpha} \delta \mathcal{A}_\nu + I_{1234}^9 \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu \\ & + I_{1234}^{10} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \delta \mathcal{A}_\nu + I_{1234}^{11} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\alpha \mathcal{H} \mathcal{D}_\mu \mathcal{H} \delta \mathcal{A}_\nu \rangle. \end{aligned} \quad (104)$$

The coefficient functions are given in Appendix C. In order to express the current in this form, partial integration has been used to remove terms of the form $\mathcal{D} \delta \mathcal{A}$. In addition, indices that are contracted with the ϵ tensor have been moved to the left, such that a term of the form $\mathcal{D}_\alpha \mathcal{D}_\mu$ yields a sum of terms of the type $\mathcal{D}_\mu \mathcal{D}_\alpha$ and $\mathcal{F}_{\mu\alpha}$.

The contributions from the variation of Eq. (100) can be grouped in three levels, with the first level having only contributions from Q and P ; the second level from the previous ones and \tilde{R} and \hat{R} ; the last level with all functions. Adding the contributions from the worldline method and the variation of Eq. (100) one obtains for the first level the following two equations

$$\begin{aligned} P_{\underline{2}1} + (m_1 + m_2) Q_{12} &= I_{12}^1, \\ (m_1 + m_2) P_{12} - (m_1^2 - m_2^2) Q_{\underline{2}1} &= I_{12}^2, \end{aligned} \quad (105)$$

which have the solution

$$Q_{12} = \frac{I_{\underline{2}1}^2}{m_1^2 - m_2^2} - \frac{P_{\underline{2}1}}{m_1 + m_2}. \quad (106)$$

Besides, there arises the following restriction which is satisfied and can serve as a check for the corresponding terms in the effective current

$$(m_1 + m_2) I_{\underline{2}1}^1 = -I_{12}^2, \quad I_{12}^1 = -I_{\underline{1}\underline{2}}^1, \quad I_{12}^2 = I_{\underline{1}\underline{2}}^2. \quad (107)$$

The matching equations for the next level are

$$-\nabla_2((m_1 + m_2)Q_{\underline{21}} - P_{\underline{21}}) + Q_{12} + Q_{\underline{21}} + (m_1 + m_3)(-\tilde{R}_{123} + \tilde{R}_{\underline{312}}) = I_{123}^3, \quad (108)$$

$$\nabla_2((m_1 + m_2)(Q_{12} + Q_{\underline{21}})) - (m_1 + m_3)\tilde{R}_{\underline{312}} - 2Q_{12} - 2Q_{\underline{21}} + \hat{R}_{\underline{312}} = I_{123}^5, \quad (109)$$

$$\begin{aligned} & \nabla_2((m_1 + m_2)P_{12}) - (m_1 - m_2)\nabla_2((m_1 + m_2)Q_{\underline{21}}) \\ & - 2P_{12} + 2(m_1 - m_2)Q_{\underline{21}} + (m_1 + m_3)(Q_{13} + 2Q_{\underline{31}}) \\ & - \hat{R}_{123}(m_1 + m_3) + (m_1 - m_2)(m_1 + m_3)\tilde{R}_{\underline{312}} = I_{123}^7, \end{aligned} \quad (110)$$

and their complex conjugates.

The first Eq. (108) is of the form

$$\tilde{R}_{123} - \tilde{R}_{\underline{312}} = -\frac{\tilde{I}_{123}^3}{m_1 + m_3}, \quad (111)$$

and a set of solutions to Eqs. (108) and (109) is hence given by

$$\tilde{R}_{123} = -\frac{1}{2} \left(\frac{\tilde{I}_{123}}{m_1 + m_3} \right)_{123} - \frac{1}{2} \left(\frac{\tilde{I}_{123}}{m_1 + m_3} \right)_{\underline{312}} - \frac{1}{2} \left(\frac{\tilde{I}_{123}}{m_1 + m_3} \right)_{\underline{231}} \quad (112)$$

$$\hat{R}_{123} = \hat{I}_{\underline{231}} - (m_1 - m_2)\tilde{R}_{123}. \quad (113)$$

The functions \tilde{I} and \hat{I} are hereby defined as

$$\tilde{I}_{123} = I_{123}^3 + \nabla_2((m_1 + m_2)Q_{\underline{21}} - P_{\underline{21}}) - Q_{12} - Q_{\underline{21}}, \quad (114)$$

$$\hat{I}_{123} = I_{123}^5 - \nabla_2((m_1 + m_2)(Q_{12} + Q_{\underline{21}})) + 2Q_{12} + 2Q_{\underline{21}}. \quad (115)$$

The last Eq. (110) leads to a constraint on the I functions that is given in Appendix C.

The function \tilde{R} possesses the required symmetries, and it reproduces the effective current correctly, but it is not necessarily finite in the coincidence limit, $m_1 \rightarrow -m_3$. One way of solving this problem is to choose the function P appropriately which up to this point remained undetermined. Such a choice is e.g. given by

$$P_{12} = \frac{I_{12}^2}{m_1 + m_2}, \quad Q_{12} = 0, \quad (116)$$

which leaves \tilde{I} as

$$\tilde{I}_{123} = I_{123}^3 - \frac{I_{\underline{21}}^2}{(m_1 - m_2)(m_2 - m_3)} + \frac{I_{\underline{31}}^2}{(m_1 - m_3)(m_2 - m_3)}. \quad (117)$$

With this choice, \tilde{R} is finite in the coincidence limit, as can be checked explicitly, and since \hat{I} is also finite, so is \hat{R} .

For the last level, the following three equations hold

$$\begin{aligned} \nabla_1 \hat{R}_{\underline{3}12} - (\nabla_2 + \nabla_3) \left((m_1 + m_3) \tilde{R}_{\underline{3}12} \right) + 2\tilde{R}_{\underline{3}12} \\ - 2(m_1 + m_4) M_{1234} = I_{1234}^9, \end{aligned} \quad (118)$$

$$\begin{aligned} (\nabla_3 - \nabla_1) \left(\tilde{R}_{\underline{3}12}(m_1 + m_3) \right) + \nabla_2 \hat{R}_{\underline{3}12} - 2\tilde{R}_{\underline{3}12} + 2\tilde{R}_{\underline{4}23} \\ - 2(m_1 + m_4) M_{\underline{4}123} = I_{1234}^{10}, \end{aligned} \quad (119)$$

$$\begin{aligned} (\nabla_1 + \nabla_2) \left((m_1 + m_3) \tilde{R}_{\underline{3}12} \right) + \nabla_3 \hat{R}_{\underline{3}12} - 2\tilde{R}_{\underline{4}23} \\ + 2(m_1 + m_4) M_{1234} = I_{1234}^{11}. \end{aligned} \quad (120)$$

One of these equations can be used to determine M , while the other two lead again to constraints on the I functions. the sum of the three equations has the especially simple form

$$-2(m_1 + m_4) M_{\underline{4}123} = -(\nabla_1 + \nabla_2 + \nabla_3) \hat{R}_{\underline{3}12} + I_{1234}^9 + I_{1234}^{10} + I_{1234}^{11}. \quad (121)$$

Since all previous functions in the effective action have been chosen finite in the coincidence limit, so is M_{1234} . Eqs. (118) and (120) show that M_{1234} is finite in the limit $m_1 \rightarrow m_2$, while Eq. (119) shows that M_{1234} is finite in the limit $m_1 \rightarrow -m_4$. This concludes the discussion of the next to leading order contributions in two dimensions.

V. CONCLUSIONS

We presented a worldline formalism to calculate the imaginary part of the covariant current of a general chiral model in the derivative expansion and our results are best summarized by Eqs. (62) and (63).

The resulting covariant current can be used to reproduce the imaginary part of the effective action by integration or matching. The advantage of this approach, compared to explicit formulas of the effective action, as given e.g. in ref. [19], consists in the manifest chiral covariance. Chiral covariance reduces the number of possible contributions to the current enormously and makes even next to leading order calculations manageable as demonstrated in section IV in the case of two dimensions. Besides the chiral covariance, the use of the worldline formalism is essential in our approach. The evaluation of the worldline path integral requires neither performing Dirac algebra nor integrating over momentum space, in contrast to the more traditional methods used in ref. [1].

In principle, it is possible to use the presented formalism to determine the effective CP violation resulting from integrating out the fermions of the Standard model. For example, in next to leading order in four dimensions, an operator of the form $\mathcal{D}\mathcal{H}\mathcal{D}\mathcal{H}\mathcal{F}\mathcal{F}$ could arise from the CP violation in the CKM matrix. Since the mass terms of the fermions are treated non-perturbatively in the derivative expansion, the resulting effective CP-violating operator is not necessarily proportional to the Jarlskog determinant. The discussion of this question is postponed to a forthcoming publication.

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APPENDIX A: INTEGRALS USED IN THE CALCULATION

In this section, the function g denotes the bosonic Green function

$$g(T, \tau_1) = \langle y(T)y(\tau_1) \rangle, \quad (\text{A1})$$

and

$$\dot{g}(T, \tau_1) = \langle \dot{y}(T)y(\tau_1) \rangle = -2\langle \psi_A(T)\psi_A(\tau_1) \rangle, \quad (\text{A2})$$

where the last expression does not contain a summation over the index A .

1. Integrals in Two Dimensions

In the calculation in two dimensions the following integrals have been used

$$\begin{aligned} J_{12}^1 &= \int_0^\infty \frac{dT}{T} \int_0^T d\tau_1 e^{-Tm_1^2 - \tau_1(m_2^2 - m_1^2)} = \frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2}, \\ J_{12}^2 &= \int_0^\infty \frac{dT}{T} \int_0^T dt_1 e^{-Tm_1^2 - \tau_1(m_2^2 - m_1^2)} \dot{g}(T, \tau_1) \\ &= -\frac{2}{m_1^2 - m_2^2} + \frac{(m_1^2 + m_2^2)}{(m_1^2 - m_2^2)^2} \log\left(\frac{m_1^2}{m_2^2}\right), \\ J_{12}^3 &= \int_0^\infty \frac{dT}{T} \int_0^T dt_1 e^{-Tm_1^2 - \tau_1(m_2^2 - m_1^2)} \dot{g}(T, \tau_1) g(T, \tau_1) \\ &= -3 \frac{m_1^2 + m_2^2}{(m_1^2 - m_2^2)^3} + \frac{(m_1^4 + 4m_1^2 m_2^2 + m_2^4)}{(m_1^2 - m_2^2)^4} \log\left(\frac{m_1^2}{m_2^2}\right). \end{aligned} \quad (\text{A3})$$

The remaining occurring integrals can be expressed as

$$\begin{aligned}
J_{12}^4 &= \int_0^\infty \frac{dT}{T} \int_0^T d\tau_1 e^{-Tm_1^2 - \tau_1(m_2^2 - m_1^2)} g(T, \tau_1) = \frac{J_{12}^2}{m_1^2 - m_2^2}, \\
J_{123}^5 &= \int_0^\infty \frac{dT}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} = -\frac{\nabla_2 J_{12}^1}{m_2 + m_3}, \\
J_{123}^6 &= \int_0^\infty \frac{dT}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_1) = -\frac{\nabla_2 J_{12}^2}{m_2 + m_3}, \\
J_{123}^7 &= \int_0^\infty \frac{dT}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_2) = -J_{321}^6. \tag{A4}
\end{aligned}$$

2. Integrals in Four Dimensions

In four dimensions the following integrals with three indices are used

$$\begin{aligned}
J_{123}^8 &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} \\
&= \frac{m_1^2 \log(m_1^2)}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)} - \frac{m_2^2(m_1^2 - m_3^2) \log(m_2^2)}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)(m_2^2 - m_3^2)} \\
&\quad + \frac{m_3^2 \log(m_3^2)}{(m_1^2 - m_3^2)(m_2^2 - m_3^2)}, \\
J_{123}^9 &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_1) \\
&= -\frac{m_1^2}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)} + \frac{m_1^2(m_1^4 - m_2^2 m_3^2) \log(m_1^2)}{(m_1^2 - m_2^2)^2(m_1^2 - m_3^2)^2} \\
&\quad - \frac{m_1^2 m_2^2 \log(m_2^2)}{(m_1^2 - m_2^2)^2(m_2^2 - m_3^2)} + \frac{m_1^2 m_3^2 \log(m_3^2)}{(m_1^2 - m_3^2)^2(m_2^2 - m_3^2)}, \\
J_{123}^{10} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-Tm_1^2 - \sum_{i=1}^2 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_2) = -J_{321}^9. \tag{A5}
\end{aligned}$$

The integrals with four indices can be expressed as

$$\begin{aligned}
J_{1234}^{11} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2 - \sum_{i=1}^3 \tau_i(m_{i+1}^2 - m_i^2)} \\
&= -\frac{\nabla_3 J_{123}^8}{m_3 + m_4}, \\
J_{1234}^{12} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2 - \sum_{i=1}^3 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_1) \\
&= -\frac{\nabla_3 J_{123}^9}{m_3 + m_4}, \\
J_{1234}^{13} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2 - \sum_{i=1}^3 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_2) \\
&= -\frac{\nabla_3 J_{123}^{10}}{m_3 + m_4}, \\
J_{1234}^{14} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2 - \sum_{i=1}^3 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_3) \\
&= -J_{4123}^{12}.
\end{aligned} \tag{A6}$$

Finally, the integrals with five indices read

$$\begin{aligned}
J_{12345}^{15} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 e^{-Tm_1^2 - \sum_{i=1}^4 \tau_i(m_{i+1}^2 - m_i^2)} \\
&= -\frac{\nabla_4 J_{1234}^{11}}{m_4 + m_5}, \\
J_{12345}^{16} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 e^{-Tm_1^2 - \sum_{i=1}^4 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_1) \\
&= -\frac{\nabla_4 J_{1234}^{12}}{m_4 + m_5}, \\
J_{12345}^{17} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 e^{-Tm_1^2 - \sum_{i=1}^4 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_2) \\
&= -\frac{\nabla_4 J_{1234}^{13}}{m_4 + m_5}, \\
J_{12345}^{18} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 e^{-Tm_1^2 - \sum_{i=1}^4 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_3) \\
&= -\frac{\nabla_4 J_{1234}^{14}}{m_4 + m_5}, \\
J_{12345}^{19} &= \int_0^\infty \frac{dT}{T^2} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 e^{-Tm_1^2 - \sum_{i=1}^4 \tau_i(m_{i+1}^2 - m_i^2)} \dot{g}(T, \tau_4) \\
&= -\frac{\nabla_3 J_{1234}^{12}}{m_3 + m_4}.
\end{aligned} \tag{A7}$$

APPENDIX B: RESULTS IN FOUR DIMENSIONS

In this section, we give the coefficient functions for the effective current and the effective density in four dimensions introduced in section III D.

The functions of the covariant current are given by

$$A_{123}^2 = \frac{m_1 m_2 - m_1 m_3 - m_2 m_3}{(m_1 + m_2)(m_1 - m_3)(m_2 - m_3)} + \frac{m_1^3(m_1(m_2 - m_3) - 2m_2 m_3) \log[m_1^2/m_3^2]}{(m_1 + m_2)(m_1 - m_3)(m_1^2 - m_2^2)(m_1^2 - m_3^2)} + \frac{m_2^3(m_2(m_3 - m_1) + 2m_1 m_3) \log[m_2^2/m_3^2]}{(m_1 + m_2)(m_2 - m_3)(m_1^2 - m_2^2)(m_2^2 - m_3^2)}, \quad (\text{B1})$$

$$A_{123}^3 = -A_{321}^1, \quad (\text{B2})$$

and

$$A_{1234}^4 = A_{1234}^R + A_{1234}^L \log[m_1^2] + A_{2341}^L \log[m_2^2] + A_{3412}^L \log[m_3^2] + A_{4123}^L \log[m_4^2], \quad (\text{B3})$$

with

$$A_{1234}^R = \frac{m_1 m_2 m_3 - m_1 m_2 m_4 + m_1 m_3 m_4 - m_2 m_3 m_4}{(m_1 - m_2)(m_1 + m_3)(m_1 - m_4)(m_2 - m_3)(m_2 + m_4)(m_3 - m_4)},$$

$$A_{1234}^L = \frac{m_1^3(-m_1^3 - m_1 m_2 m_3 + m_1 m_2 m_4 - m_1 m_3 m_4 + 2m_2 m_3 m_4)}{(m_1 - m_2)(m_1 + m_3)(m_1 - m_4)(m_1^2 - m_2^2)(m_1^2 - m_3^2)(m_1^2 - m_4^2)}. \quad (\text{B4})$$

The explicit functions occurring in the effective density are rather lengthy and hence we display them in terms of the integrals presented in the last section.

$$B_{123}^3 = J_{123}^8(m_1 + m_5) - J_{123}^9(m_1 - m_3) - J_{123}^{10}(m_2 - m_3),$$

$$B_{1234}^4 = J_{1234}^{11}(m_1 + m_4) - J_{1234}^{12}(m_1 - m_2) - J_{1234}^{13}(m_2 + m_3) + J_{1234}^{14}(m_4 + m_3),$$

$$B_{1234}^5 = J_{1234}^{11}(m_1 + m_4) - J_{1234}^{12}(m_1 + m_2) - J_{1234}^{13}(m_2 - m_3) + J_{1234}^{14}(m_4 + m_3),$$

$$B_{1234}^6 = B_{4321}^4,$$

$$B_{12345}^7 = J_{12345}^{15}(m_1 + m_5) - J_{12345}^{16}(m_1 + m_2) + J_{12345}^{17}(m_2 + m_3) - J_{12345}^{18}(m_3 + m_4) + J_{12345}^{19}(m_4 + m_5). \quad (\text{B5})$$

APPENDIX C: COVARIANT CURRENT IN NEXT TO LEADING ORDER

In this section we summarize the contributions to the covariant current in next to leading order. For the first two levels, they are given by

$$\begin{aligned}
I_{12}^1 &= \frac{I_{21}^2}{m_1 - m_2}, \quad I_{12}^2 = -4(m_1^2 - m_2^2)J_{12}^3 + 4(m_1 + m_2)^2 J_{12}^4, \\
I_{123}^3 &= \frac{m_1 - m_2 + 2m_3}{m_2^2 - m_3^2} \nabla_2(I_{21}^2) + 8 \frac{m_1 + m_3}{m_2^2 - m_3^2} \nabla_2(m_1 m_2 J_{12}^4) \\
&\quad + 16 \frac{m_1 m_3 J_{13}^4}{m_2^2 - m_3^2} - \frac{I_{31}^2}{(m_1 - m_3)(m_2 - m_3)}, \\
I_{123}^5 &= 2 \frac{\nabla_2(I_{21}^2)}{m_1 - m_2} + 3 \frac{I_{31}^2}{(m_1 - m_3)(m_1 - m_2)} - I_{123}^3 - \frac{I_{132}^7}{m_1 - m_2}, \\
I_{123}^7 &= (m_2 + m_3) \left(\left(\frac{\nabla_1 I_{12}^2}{m_1 + m_2} \right)_{\underline{312}} - \frac{\nabla_1 I_{12}^2}{m_1 + m_2} \right) + \frac{3m_2 - m_3}{m_2 + m_3} \nabla_2 I_{12}^2 \\
&\quad + 4(m_2 + m_3)(m_1(m_1 - m_2 - m_3) + m_2 m_3) \left(\left(\frac{\nabla_1 J_{12}^4}{m_1 + m_2} \right)_{\underline{312}} - \frac{\nabla_1 J_{12}^4}{m_1 + m_2} \right) \\
&\quad - 8(m_2 + m_3) \left(\left(\frac{\nabla_1(m_1 m_2 J_{12}^4)}{m_1 + m_2} \right)_{\underline{312}} - \frac{\nabla_1(m_1 m_2 J_{12}^4)}{m_1 + m_2} \right) \\
&\quad + 4(m_2 - m_3) \left(\frac{\nabla_2((m_1^2 - 4m_1 m_2 + m_2^2)J_{12}^4)}{m_2 + m_3} - \nabla_2((m_1 - m_2)J_{12}^4) \right), \tag{C1}
\end{aligned}$$

and the relations

$$I_{123}^4 = -I_{321}^3, \quad I_{123}^6 = -I_{321}^5, \quad I_{123}^8 = I_{321}^7. \tag{C2}$$

The contributions to the last level read

$$\begin{aligned}
I_{1234}^9 &= -I_{4321}^{11}, \\
I_{1234}^{10} &= -\frac{m_1 + m_4}{m_3 - m_4} I_{4123}^{11} + \frac{1}{2} \left(\nabla_2 + \frac{m_1 + m_4}{m_3 - m_4} (\nabla_3)_{4123} \right) I_{123}^5 \\
&\quad + \left(\nabla_2 + \nabla_3 + \frac{m_1 + m_4}{m_3 - m_4} (\nabla_3)_{4123} \right) \frac{m_1 + m_3}{m_2 - m_3} (I_{213}^3 - (\nabla_2 I_{12}^1)_{213}) \\
&\quad + \frac{1}{2} \left(\nabla_2 + \nabla_3 + \frac{m_1 + m_4}{m_3 - m_4} (-\nabla_1 + \nabla_2 + \nabla_3)_{4123} \right) (I_{123}^3 - \nabla_2 I_{12}^1) \\
&\quad + \frac{1}{2} \left(-\nabla_1 + \nabla_2 + \frac{m_1 + m_4}{m_3 - m_4} (\nabla_3)_{4123} \right) (I_{321}^3 - (\nabla_2 I_{12}^1)_{321}) \\
&\quad - \frac{m_1 + m_3}{2(m_2 - m_3)(m_3 - m_4)} (I_{213}^3 - (\nabla_2 I_{12}^1)_{213}) + \frac{2}{m_2 + m_4} (I_{432}^3 - (\nabla_2 I_{12}^1)_{432}) \\
&\quad - \frac{1}{2(m_3 - m_4)} (I_{421}^3 - (\nabla_2 I_{12}^1)_{421}), \\
I_{1234}^{11} &= \nabla_2 I^4 - \frac{I_{124}^4}{m_3 - m_4} - \frac{4(m_1^2 - m_3^2)J_{13}^3}{(m_3 - m_4)(m_1^2 - m_4^2)} - \frac{4(m_3 + m_4)J_{34}^4}{m_1^2 - m_4^2} - \frac{4J_{124}^5}{m_3 - m_4} \\
&\quad - \frac{m_1 + m_2}{m_3 - m_4} \left(\nabla_2 - \frac{m_2 + m_3}{m_3 + m_4} \nabla_3 \right) \left(\frac{\nabla_1((m_1^2 - m_2^2)(I_{12}^2 - 8m_1 m_2 J_{12}^4))}{(m_1^2 - m_2^2)(m_1 + m_2)} \right) \\
&\quad + \frac{(m_3 + m_4)(\nabla_1(I_{12}^2 - 8m_1 m_2 J_{12}^4))_{413}}{(m_1^2 - m_4^2)(m_1 - m_4)} + \frac{4(m_2 + m_3)(m_1 - m_4)}{m_1 + m_2)(m_3 - m_4)} \nabla_1 J^4 \\
&\quad - \frac{4(m_1 + m_2)}{m_3 - m_4} \left(-\nabla_2 + \frac{m_2 + m_3}{m_3 + m_4} \nabla_3 \right) \frac{(m_1^2 + m_3^2)J_{13}^4 - (m_2^2 - m_3^2)J_{23}^4}{m_1^2 - m_2^2} \\
&\quad - \frac{4(m_2 + m_3)(m_1 - m_4)(m_3 + m_4)}{m_1 + m_2} \nabla_1 \frac{(\nabla_1 J^4)_{312}}{m_1 - m_3} - \frac{4(m_1 + m_3)}{m_3 + m_4} \nabla_3 J^5 \\
&\quad - \frac{4(m_1 + m_2)(m_2 + m_4)}{m_3^2 - m_4^2} \nabla_3 J^6. \tag{C3}
\end{aligned}$$

All functions I are finite in the coincidence limit and must fulfill the following constraints due to the behavior of the terms in the effective action under cyclic permutation and complex

conjugation:

$$\begin{aligned}
& \frac{(m_1 + m_3)I_{312}^3}{m_2 - m_3} + I_{321}^3 + \frac{I_{13}^2}{(m_1 - m_2)(m_2 - m_3)} + \frac{I_{31}^2}{(m_1 - m_2)(m_1 - m_3)} \\
& - \frac{(m_1 + m_3)I_{23}^2}{(m_1 - m_2)(m_2^2 - m_3^2)} - \frac{I_{32}^2}{(m_1 - m_2)(m_2 - m_3)} = 0, \\
& I_{123}^3 + I_{321}^3 - \frac{(m_1 + m_3)I_{231}^3}{m_1 - m_2} + I_{123}^5 + I_{321}^5 \\
& + \frac{I_{12}^2}{m_1^2 - m_2^2} + \frac{I_{21}^2}{(m_1 - m_2)(m_2 - m_3)} - \frac{I_{31}^2}{(m_1 - m_2)(m_2 - m_3)} = 0, \\
& (m_1 + m_3)I_{312}^3 + (m_1 + m_3)I_{312}^5 - I_{123}^8 + \frac{2I_{13}^2}{m_1 - m_2} \\
& + \frac{(3m_1 - m_2 + 2m_3)I_{23}^2}{(m_1 - m_2)(m_2 + m_3)} = 0, \quad (C4)
\end{aligned}$$

and

$$I_{123}^3 = I_{123}^3 = 0, \quad I_{123}^5 = I_{123}^5 = 0, \quad I_{123}^7 = -I_{123}^7 = 0. \quad (C5)$$

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